CHAPTER 1

Analyzing Mathematics Instructional Tasks

Mathematical tasks can be examined from a variety of perspectives, including the number and kinds of representations evoked, the variety of ways in which they can be solved, and their requirements for student communication. In this book, we examine mathematical instructional tasks in terms of their cognitive demands. By cognitive demands we mean the kind and level of thinking required of students in order to successfully engage with and solve the task.

In this chapter, we describe a method for analyzing the cognitive demands of tasks as they appear in curricular or instructional materials (the first phase of the Mathematical Tasks Framework shown in Figure I.3 in the Introduction). Unlike the remainder of the framework, which describes task evolution during a classroom lesson, the initial phase of the framework focuses on tasks before the lesson begins, that is, the task as it appears in print form or as it is created by the teacher.

Why are the cognitive demands of tasks so important? Opportunities for student learning are not created simply by putting students into groups, by placing manipulatives in front of them, or by handing them a calculator. Rather, it is the level and kind of thinking in which students engage that determines what they will learn (NCTM, 1991). Tasks that require students to perform a memorized procedure in a routine manner lead to one type of opportunity for student thinking; tasks that demand engagement with concepts and that stimulate students to make purposeful connections to meaning or relevant mathematical ideas lead to a different set of opportunities for student thinking. Day-in and day-out, the cumulative effect of students’ experiences with instructional tasks is students’ implicit development of ideas about the nature of mathematics—about whether mathematics is something they personally can make sense of, and how long and how hard they should have to work to do so.

Since the tasks with which students become engaged in the classroom form the basis of their opportunities for learning mathematics, it is important to be clear about one’s goals for student learning. Once learning goals for students have been clearly articulated, tasks can be selected or created to match
The example shown in Figure 1.1 illustrates four ways in which students can be asked to think about the relationships among different representations of fractional quantities. Each of these ways places a different level of cognitive demand on students. Low-level demands on students would consist of memorizing the equivalent form of specific fractional quantities (e.g., $\frac{1}{4} = 0.25 = 25\%$) or performing conversions of fractions to percent or decimals using standard conversion algorithms in the absence of additional context or meaning (e.g., convert the fraction $\frac{9}{4}$ to a decimal by dividing the numerator by the denominator to get $3.75$; change $3.75$ to a percent by multiplying by $100$ to get $375\%$). These low-level tasks are classified as memorizing or procedures without connections, respectively. When tasks such as these are simply used to highlight students' ability to perform standard algorithms, it is important that the teacher help the students see the context or meaning behind the procedures they are using (NCTM, 1991). The table below shows how students might be asked to identify the decimal and percent equivalents for the fraction $\frac{1}{4}$ and $\frac{1}{2}$, and then use these conversions to solve a problem involving ratios and proportions.

<table>
<thead>
<tr>
<th>Fraction</th>
<th>Decimal</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{4}$</td>
<td>0.25</td>
<td>25%</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>0.50</td>
<td>50%</td>
</tr>
</tbody>
</table>

The higher-level demands on students would consist of understanding the underlying connections among equivalent representations of fractions, decimals, and percents. For example, students might be asked to use the fraction $\frac{3}{4}$ to identify the decimal and percent equivalents of $75\%$. These higher-level tasks are classified as understanding and procedures with connections (NCTM, 1991). The table below shows how students might be asked to identify the decimal and percent equivalents for the fraction $\frac{3}{4}$, and then use these conversions to solve a problem involving ratios and proportions.

<table>
<thead>
<tr>
<th>Fraction</th>
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<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{3}{4}$</td>
<td>0.75</td>
<td>75%</td>
</tr>
</tbody>
</table>

Doing Mathematics

Shade 6 small squares in a 4 x 10 rectangle. Using the rectangle, explain how to determine each of the following: a) the percent of area that is shaded, b) the decimal part of area that is shaded, and c) the fractional part of area that is shaded.

One Possible Student Response:

- a) One column will be $\frac{1}{10}$ since there are 10 columns. The second column has 2 squares shaded so that would be one half of $\frac{1}{10}$, which is $\frac{1}{20}$ or 0.05. So the 6 shaded blocks equal 0.3 plus 0.05, which equals 0.35.
- b) Six shaded blocks out of 40 squares is $\frac{6}{40}$, which reduces to $\frac{3}{20}$.
- c) Six shaded squares out of 40 squares is $\frac{6}{40}$.
Another high-level task (classified as *doing mathematics*) would entail asking students to explore the relationships among the various ways of representing fractional quantities. Students would not—at least initially—be provided with the conventional conversion procedures. They might once again use grids, but this time grids of varying sizes (not just 10 × 10) would be used. As shown in Figure 1.1, students could be asked to shade six squares of a 1 × 1 rectangle and to represent the shaded area as a percent, a decimal, and a fraction. When students use the visual diagram to solve this problem, they are challenged to apply their understandings of the fraction, decimal, and percent concepts in novel ways. For example, once a student has shaded the six squares, he or she must determine how the six squares relate to the total number of squares in the rectangle. In Figure 1.1, we see an example of a student’s response to this task that illustrates the kind of mathematical reasoning used to come up with an answer that makes sense and that can be justified. In contrast to the tasks with lower-level demands discussed earlier, in *procedures-with-connections or doing-mathematics* tasks, students typically perform far fewer problems (sometimes as few as two or three) in one sitting.

**MATCHING TASKS WITH GOALS FOR STUDENT LEARNING**

As illustrated by the above discussion, not all mathematical tasks provide the same opportunities for student learning. Some tasks have the potential to engage students in complex forms of thinking and reasoning while others focus on memorization or the use of rules or procedures. In our work with teachers in the QUASAR Project, we discovered the importance of matching tasks with goals for student learning. Take, for example, the case of Mr. Johnson (Silver & Smith, 1996). Mr. Johnson wanted his students to learn to work collaboratively, to discuss alternative approaches to solving tasks, and to justify their solutions. However, the tasks he tended to use (e.g., expressing ratios such as $\frac{15}{24}$ in lowest terms) provided little, if any, opportunity for collaboration, exploration of multiple solution strategies, or meaningful justification. Not surprisingly, class discussions were not very rich or enlightening. The discourse focused on correct answers and describing procedures, doing little to further students’ ability to think or reason about important ideas associated with ratio and proportion.

Mr. Johnson’s experience (and that of many teachers with whom we have worked) makes clear the need to start with a cognitively challenging task that has the potential to engage students in complex forms of thinking if the goal is to increase students’ ability to think, reason, and solve problems. Although starting with such a task does not guarantee student engagement at a high level, it appears to be a necessary condition since low-level tasks rarely result in high-level engagement (Stein, Grover, & Henningsen, 1996).

This is not to suggest that all tasks used by a teacher should engage students in cognitively demanding activity, since there may be some occasions on which a teacher might have other goals for a particular lesson, goals that would be better served by a different kind of task. For example, if the goal is to increase students’ fluency in retrieving basic facts, definitions, and rules, then tasks that focus on *memorization* may be appropriate. If the goal is to increase students’ speed and accuracy in solving routine problems, then tasks that focus on *procedures without connections* may be appropriate. Use of these types of tasks may improve student performance on tests that consist of low-level items and may lead to greater efficiency of time and effort in solving routine aspects of problems that are embedded in more complex tasks. However, focusing exclusively on tasks of these types can lead to a limited understanding of what mathematics is and how one does it. In addition, an overreliance on these types of tasks could lead to the inability to apply rules and procedures more generally, that is, to similar but not identical situations, or to recognize whether a particular rule or procedure is appropriate across a variety of situations (NCTM, 1989; 2000). Hence, students also need opportunities on a regular basis to engage with tasks that lead to deeper, more generative understandings regarding the nature of mathematical processes, concepts, and relationships.

**DIFFERENTIATING LEVELS OF COGNITIVE DEMAND**

The Task Analysis Guide (shown in Figure 1.2) consists of a listing of the characteristics of tasks at each of the levels of cognitive demand described earlier in the chapter: *memorization, procedures without connections, procedures with connections,* and *doing mathematics.* When applied to a mathematical task (in print form), this guide can serve as a judgment template (a kind of scoring rubric) that permits a “rating” of the task based on the kind of thinking it demands of students.

For example, the guide would be helpful in deciding that the Fencing Task (shown in Figure 1.2 in the Introduction) was an example of *doing mathematics* since the characteristics of this level most clearly describe the kind of thinking required to successfully complete the task. Specifically, no pathway is suggested by the task (i.e., there is no overarching procedure or rule that can simply be applied for solving the entire problem and the sequence of necessary steps is unspecified) and it requires students to explore pens of different dimensions and ultimately to make a generalization regarding the pen that will have maximum area for a fixed amount of fencing.
When determining the level of cognitive demand provided by a mathematical task, it is important not to become distracted by superficial features of the task and to keep in mind the students for whom the task is intended. Both of these considerations are discussed below.

**Going Beyond Superficial Features**

Determining the level of cognitive demand of a task can be tricky at times, since superficial features of tasks can be misleading. Low-level tasks, for example, can appear to be high-level when they have characteristics of reform-oriented instructional tasks (NCTM, 1991; Stein et al., 1996), such as requiring the use of manipulatives; using "real-world" contexts; involving multiple steps, actions, or judgments; and/or making use of diagrams. For example, some individuals have considered Martha's Carpentry Task (shown in Figure 1 in the Introduction) a high-level task because it is a word problem and it is set in a real-world context. Similarly, some have considered commonly used fraction tasks—which ask students to find the sum of two proper fractions with unlike denominators and then to show the answer using fraction strips—high-level because they use manipulatives. But we would classify these tasks as low-level because typically well-rehearsed procedures (for Martha's Carpentry, the formula for determining area, and for the fraction task, the rule for adding fractions with unlike denominators) are strongly implied by the problems. In both cases, the tasks would be considered to be procedures-without-connections tasks since there is little ambiguity about what has to be done or how to do it, there is no connection to concepts or meaning required, and the focus is on producing the correct answer.

It is also possible for tasks to be designated low-level when in fact they should be considered high-level. For example, the Lemonade Task—in which students have to determine which of two recipes for lemonade is more "lemony": Recipe A, which has 2 cups of lemon concentrate and 3 cups of water; or Recipe B, which has 3 cups of lemon concentrate and 5 cups of water—has been considered by some an example of a procedures-without-connections task because it "looks like" a standard textbook problem that could be solved by applying a rule or because it lacks "reform features" (such as requiring an explanation or justification). However, we have described this task as doing mathematics since no pathway for solving the problem is suggested (either explicitly or implicitly). Specifically, the task requires students to compare two situations and to determine which recipe has the higher proportion of concentrate. To do so, students must make sense of the problem situation and maintain a close connection to the meaning of ratio and to the question being asked. So even