The Teacher’s Role in Teaching Mathematics through Problem Solving

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A few secondary school students can learn mathematics and become expert problem solvers through independent study and by solving many mathematical problems. Very few. For the vast majority of students, becoming mathematically proficient and developing problem-solving expertise involve studying mathematics under the guidance of a well-qualified teacher who selects problems that embody important mathematical ideas and then ensures that the mathematics involved in these problems is brought to the surface and made explicit. This chapter identifies and discusses teaching strategies that have the potential to help students learn important mathematical ideas as they work on significant mathematical tasks and move toward becoming mathematically literate. Following the organizational framework used by Charles and Lester (1982), I examine the teacher’s role in instruction before, during, and after lessons designed to teach mathematics through a problem-solving approach.

The Teacher’s Role

The teacher’s role in fostering students’ mathematical learning is central and deserves greater attention than it has received in recent years. Does effective mathematics teaching involve more than forming small groups and assigning good problems for students to solve? Of course it does, and the purpose of this chapter is to delve into the many decisions that teachers make that influence the quality of the mathematics their students learn. The
teacher's role in directing the development of students' mathematical knowledge is indeed complex. The responsibility is difficult to fulfill because it involves simultaneously fostering students' knowledge accumulation and developing their mathematical abilities. The mathematical proficiency we seek requires depth of knowledge and includes the strands of conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (National Research Council 2001). A teacher must bring all these components together in lessons while simultaneously taking into account students' abilities and backgrounds. No wonder knowledgeable people proclaim that teaching mathematics is one of the most demanding professions.

Before the Lesson

When one watches a great mathematics lesson, the "behind the scenes" time and effort that have been devoted to planning and perfecting the lesson are not often evident. The impact of a lesson on students' learning is greatly influenced by more than what one can observe in watching a lesson. For example, careful attention to the mathematical ideas to be taught is essential in preparing the lesson, and such attention to detail has been shown to be associated with high achievement on the part of students (Stigler and Hiebert 1999). We begin by examining how attention can be given to mathematics content as we consider some of the important decisions that teachers make prior to teaching a lesson that promotes mathematical proficiency. Of necessity, I discuss these teaching decisions one at a time, understanding that all aspects of teaching are deeply interconnected. Teaching decisions in one area must take into account the entire teaching context because changing one instructional component always affects, directly or indirectly, other components.

Using Tasks to Structure Lessons

Teaching mathematics through problem solving has been described carefully in preceding chapters. In reading this chapter, an important point to remember is that successfully implementing such an approach involves many teacher decisions and actions, which include, to name a few, choosing appropriate tasks, conveying tasks to students in ways that stimulate interest, maintaining students' engagement in tasks, and leading discussions in which the important mathematical ideas embedded in the tasks are brought to the surface. As we consider these teaching decisions and roles, we arrive at the clear conclusion that teaching through problem solving is quite different from other activities that some-

times masquerade as such, including discussing problems-of-the-week, playing strategy games, working on logic puzzles, and focusing on application-type problems at the end of textbook chapters. One of the most important distinctions between teaching through problem solving and teaching that simply involves using problems is how tasks are chosen.

Choosing Appropriate Tasks

Mathematics educators usually agree that the mathematical tasks teachers choose to use in their teaching affect not only the mathematics that students learn but also the depth and quality of that learning. In fact, some research evidence shows that using mathematically rich tasks in the classroom, even though used poorly, can have positive effects on students' learning (Stein and Lane 1996).

Teachers can select tasks that develop important aspects of mathematical proficiency. These components include the learning of mathematical concepts (e.g., prime numbers), skills (e.g., computational estimation), problem-solving techniques (e.g., "guess and check"), methods of reasoning (e.g., pigeonhole principle), and proof techniques (e.g., exhaustion), to name a few. Elaborated examples of how carefully chosen tasks can lead to the preceding learning outcomes can be found in other chapters of this volume. A reasoning example is presented here.

Reasoning example

The importance of developing mathematical reasoning is touted in most, if not all, recent recommendations for school mathematics (e.g., NCTM 2000; National Research Council 2001). As we shall see later, mathematical reasoning can be promoted through appropriate teacher questioning when discussing a wide variety of rich mathematical tasks. Many useful reasoning strategies exist that students should acquire, and teachers can make progress toward this goal by carefully selecting tasks in which to engage students. Consider the pigeonhole reasoning strategy, for example. Simply stated, it says that if we have a finite number of holes, say, \( n \), and we place \( n + 1 \) pigeons in these holes, then we will have a hole with at least two pigeons in it. The strategy has applicability to a wide variety of mathematical problems and situations ranging from the simple to the complex, and from arithmetic to probability to abstract algebra. By carefully choosing an appropriate task, a teacher can help students learn to use the strategy and develop other mathematical ideas at the same time. Consider the following examples chosen from different areas of mathematics:
**Probability task**

What is the minimum number of times you have to roll a fair die before you can be certain that you will get an outcome that is a repeat of a previous roll?

*One solution.* Think of each of the six outcomes possible on a single roll of the die as your pigeonholes \( (n = 6) \); then from the pigeonhole principle, by the seventh roll you will always have a repeated outcome (i.e., two pigeons in the same pigeonhole). Note that one can easily show by example that any number of rolls less than seven will not satisfy the problem conditions.

**Number theory task**

Given any nine integers, show that you can choose two of them whose difference is a multiple of 8.

*One solution.* The reader is encouraged to try to find the solution before reading on. Notice that any number divided by 8 has one of eight remainders, 0 through 7. These remainders are our eight pigeonholes. The nine given numbers are our pigeons. By the pigeonhole principle, two of the numbers are in the same hole and thus have the same remainder when divided by 8, say, \( r \). Thus, through some symbolic representation and algebraic manipulation, we can show that the two numbers can be expressed as \( 8m + r \) and \( 8n + r \), with \( m \) and \( n \) being integers. Their difference is 8 \( (m-n) \), which is divisible by 8.

In choosing tasks that are appropriate and productive, some general learning principles are helpful to keep in mind:

- The tasks presented to students affect the mathematical concepts and skills that students are likely to acquire as they complete the task.

On the one hand, one cannot predict with precision the exact learning path a student will follow during the course of a lesson or in working with a task; yet one can be quite confident that students will interact with a specific mathematical idea if the task is carefully selected to embody that idea, as shown in the preceding examples. On the other hand, if one chooses a task without regard to a given mathematical objective, then the likelihood of a student's learning something about that objective is remote.

- The tasks presented to students affect the problem-solving techniques and habits of mind that students learn.

Students should acquire many problem-solving techniques as part of the secondary school mathematics curriculum. Giving explicit attention to these techniques in lesson planning is appropriate. Familiar techniques that quickly come to mind include making an organized list, guessing and checking, using the method of exhaustion, applying the pigeonhole principle, and making a diagram. A careful choice of tasks allows students to explore and learn a particular problem-solving technique as part of a lesson. The same can be said for habits of mind (see Levavasseur and Cuoco [this volume]).

- The tasks presented to students should require a high level of cognitive demand to promote the development of a deep knowledge of mathematics.

What do I mean by high cognitive demand in tasks? One way to think about this idea is simply to consider the amount of thinking and sense making needed to complete the task. If the task focuses primarily on recall or on simple application of a learned procedure, then it is at the lower end of a continuum of cognitive demand. But if the task requires using connections between ideas, or making new connections, or working in a new context, or putting several procedures and concepts together in a new way, or some combination of these approaches, then the task is at the upper end of the continuum. Clearly, a task with a high cognitive demand will not be completed quickly nor have an immediately obvious solution path.

Teachers have numerous good reasons to strive to include tasks that embody a high cognitive demand in their lessons. First, using these tasks allows students to engage in "doing" mathematics and thus gives them opportunities to develop the capacity to think and reason mathematically. Second, research evidence shows that even when teachers begin with high-demand tasks, the tasks have a tendency to decline in cognitive demand as they are implemented in the classroom (Smith 2000; Stein, Grover, and Henningsen 1996). Thus, starting with high-level tasks becomes even more important. Finally, when students engage in tasks that go beyond memorizing facts and applying known formulas and procedures, they develop a better sense of what mathematics is, that is, a discipline in which the ideas make sense, are logically related to one another, and can be used to solve nontrivial problems.

Another way of thinking about choosing tasks is to focus on selection criteria. The following criteria, some of which are discussed in more detail by Marcus and Fey in this volume, are important in choosing good tasks. A good task generally has the following characteristics:
• It is centered on an important mathematical idea, concept, or skill that is part of a course of study.

• It is clearly stated, that is, it is not unintentionally vague or misleading.

• It involves an important real-world context or mathematical context that has the potential to attract and maintain students’ interest.

• It can be solved with a range of methods from informal to more formal. Its solution or solutions can be productively approached at several levels of sophistication.

• It has interesting extensions. Most rich tasks have interesting extensions, and these extensions are valuable for at least three reasons. First, they help the teacher accommodate individual differences within the classroom. Simpler versions of the task can be assigned to students having difficulties, and more complex extensions can be used to challenge high-achieving students who finish the initial task quickly. Second, allowing students to suggest extensions to tasks is a nice segue to problem posing, a process that has been associated with increasing students’ learning. (See Goldenberg and Walter [this volume] for more information on problem posing.) Third, looking at problem extensions can often lead to students’ making and verifying mathematical generalizations, two important skills that mathematically proficient students should have.

Forming Good Questions

A good question captures one’s interest, starts mental activity, and may stimulate creative thinking. Given this potential, a teacher must carefully form some appropriate questions in advance as part of lesson preparation. In this preparation, he or she should pay particular attention to generating questions that promote students’ ability to make generalizations, that provide hints for making progress on a task without giving away the solution path, and that stimulate higher-order thinking. These types of questions are the most difficult to form in the midst of a class session, yet they have great potential to enhance students’ learning.

Class Organization and Lesson Structure

One of the decisions a teacher makes in planning a lesson is how the class will be organized for the lesson. Many choices and combinations of choices are available. Certainly the most popular lesson format in secondary school mathematics classrooms is the whole-class arrangement. However, having students work in small groups has been used increasingly in recent years, and other formats, such as student presentations and independent study, are used regularly by some teachers for either an entire lesson or part of a lesson. Any of these formats can be used successfully; their effectiveness depends, in large part, on how they are used. Because small-group instruction has been strongly advocated for greater use by many mathematicians educators and because it has been used inappropriately in some situations, I give it special consideration in the discussion that follows. The focus on it should not be taken as a signal that it is the preferred method of teaching mathematics through problem solving.

The Role of Small Groups in Problem Solving

Including small-group work when teaching through a problem-solving approach can facilitate mathematical discussions. Small-group work presents opportunities for more students to verbalize their questions and thinking than whole-class instruction does. Further, for some students, participating in small groups may be less intimidating than engaging in whole-class discussions. Two important principles should be considered when making decisions about the use of small groups.

• Not every problem or task lends itself to small-group work. In general, for a problem having one primary solution path and a single solution, small-group work may degenerate to a situation in which one student in the group—often, but not always, the highest achieving student—solves the problem and explains it to the other students. Teachers can reduce the likelihood of this occurrence by using a variety of techniques (see, e.g., Arzt and Newman [1997]; Davidson and Worsham [1992]), including assigning unique roles for each student to assume within the group or insisting that every student in the group must be able to explain the group’s solution to the task. When selecting tasks for small-group work, however, teachers can select from many rich tasks and so have no need to limit problems to those that have a predominant solution path and a single correct answer.

• Tasks that have a variety of reasonably apparent, viable solution methods and multiple solutions are good task choices for group work. Such tasks allow students to pursue solutions in different ways and then to describe their solution paths to their peers. This process demonstrates
that generally many ways are available to solve a mathematical problem and provides practice in communicating orally about mathematics. Further, because multiple solutions to the task are possible, different students can contribute different parts to the total solution.

In closing this section, a point worth mentioning again is that what a teacher does within a chosen lesson organization is particularly important, and we discuss this role and these decisions next.

During the Lesson

Teachers make literally thousands of decisions each day as they teach their lessons (Good and Brophy 1996). They make some of the more important decisions early in the lesson when the tasks for the day are conveyed to students and they begin work. Although one cannot prescribe how to present a task, benchmarks can provide a useful basis for successful presentations.

Presenting Tasks

Obviously, an interesting task will motivate students to begin work on it. To make significant progress toward solving a problem and learning the mathematics within it, however, requires that students come to fully understand the task. that is, to know what is given and what is to be found or shown in more than a superficial way. When difficulties arise at this introductory stage and the teacher senses that students are beginning to lose interest, she or he can often help the students refocus on the task by asking direct questions about what is given and what is to be found. This questioning may include asking students to tell the meaning of various terms and phrases in the task or modeling the problem in some way using either symbols or actions. In the latter instance, the teacher is wise to draw attention initially to problem conditions and not to potential solution methods or paths.

One presentation strategy that some teachers have found effective is presenting a task in several parts, with early parts relatively easy to solve and subsequent parts more challenging. See Marcus and Fey in this volume for examples that use this strategy. This teaching strategy also attends to individual differences among students in the class by allowing slow workers sufficient time to complete at least the first part of the task and by providing high-achieving students with engaging work to do after they quickly finish the first part of the task. At some point in a lesson, a discussion of (a) the task, (b) its various solution paths, and (c) valid solutions is essential to ensure that the mathematics embedded within the task is brought to the surface and that every student develops an understanding of it. Questioning plays an important role in these culminating discussions.

Using Questions

Educators generally agree that students learn best from instruction when they are actively involved, that is, when they are mentally interacting with the ideas being considered. Asking questions is one way to promote intellectual involvement. Although questions asked by students are an important and productive part of every successful mathematics lesson, a teacher with a deep and thorough understanding of mathematics can pose questions that not only foster students' involvement but also lead directly to the development of important mathematical ideas and understandings—the "residue" ideas.

Some guidelines for teachers as they ask questions are the following:

- Encourage reasoned guessing, not savage guessing (Pólya 1945).
- Establish the norm that students' responses should include a rationale (Mason, Burton, and Stacey 1982).
- Always be aware of who is doing the thinking, the teacher or the student.

Teachers' questions that encourage students' thinking are especially important in conducting good lessons. The following questions are some that can be used to promote students' thinking:

- How did you decide on a solution method to try?
- How did you solve the problem?
- Did anyone solve it in a different way?
- How would you compare these solution methods?
- Which of the solution methods do you like best? Why?
- Can you tell me how you solved the problem without saying the answer?
- Does this problem remind you of any other problems you have solved?
- How can we change the problem to get another interesting problem?
What mistakes do you think some students might make in solving this problem?

Clearly the preceding list of suggested questions is far from exhaustive, but it is indicative of the type of open-ended questions that cause students to think about, and reflect on, mathematical ideas.

Creating Classroom Climate and Developing Mathematical Dispositions

Classroom climate is shaped largely by the decisions a teacher makes about many lesson components, including the mathematical tasks considered, the amount of time devoted to investigation, the value placed on discussion, the treatment of incorrect answers and errors in reasoning, the placement of authority with regard to correctness, and so on. The teacher directly or indirectly controls all these factors and thus is a major determen of classroom climate. I discuss a limited number of aspects of classroom climate here. See Rasmussen, Yackel, and King in this volume for a detailed discussion of social norms and classroom climate.

The way teachers react to students' responses and questions has a major influence on classroom climate. And classroom climate affects students' willingness to respond to questions, describe their thinking, share their solution methods, posit hypotheses, suggest generalizations, and, in general, participate in "doing" mathematics in the classroom. What kind of classroom climate is needed? Certainly the atmosphere to seek is one in which effort is valued and mistakes are recognized as a means to learn. An emphasis on giving reasons for responses and a focus on making sense of the mathematics are also necessary. As one would expect, developing such a classroom climate takes time. We emphasize here that how a teacher acts and reacts in front of a class communicates to students important notions about how mathematics is done and about the nature of mathematics.

Consider the following three ideas:

- How a teacher solves a problem in front of the class is crucial. A teacher must do some acting as he or she solves a task for the class, as Pólya (1945) pointed out long ago. Because the task presented is often an exercise for the teacher, how she acts out solving it influences greatly what students think about how problems are solved in mathematics. As we all know, solving a mathematics problem involves advances and retreats, and moments of frustration and excitement, to name but a few of the cognitive and emo-

tional components. A teacher should communicate these components to students while demonstrating a solution to a task, teaching students not always to expect a smooth march to a mathematical problem's solution. Accepting that difficulties are normal and that they should not be a cause for distress or quitting is a lesson that can have positive long-term effects. For this reason alone, teachers need to regularly solve problems in front of the class.

- A teacher should handle inappropriate solution methods and incorrect solutions that students bring forward in a way that conveys the idea that mathematics must make sense. That is, often the teacher can discuss an incorrect method or solution in a way that pinpoints a student's false assumption by showing that had the assumption been true, then the student's proposed solution method or answer would have been correct. Dealing with a student's idea in this respectful way casts a positive light on the student's response and is far preferable to a response that dismisses the proposed method or solution out of hand.

- Through questioning and modeling solutions, teachers can help students develop habits of mind that will be useful to them in all their mathematical studies. Consider the following questions: (1) Is the answer reasonable? (2) Did I recheck the steps taken to find the solution? (3) Is this problem related to other problems? (4) Can I change the problem to make it easier to solve? (5) Is another solution possible? Asking these questions of students or mentioning them aloud when solving a problem in front of the class suggests that these questions are reasonable for anyone to ask when solving a problem, possibly fostering their use by students whenever they confront a mathematical task or situation. In general, some of the attributes of classroom interaction that facilitate the development of a positive learning climate and desirable habits of mind are these:

  > Learning mathematics as sense-making prevails.
  > Reasonable guessing, as opposed to savage guessing, is valued.
  > Sharing one's thinking is expected.
  > Justifying one's assertions is the norm.
The Teacher's Role in Teaching Mathematics

Focusing on Students' Thinking

The means available for assessing students' thinking generally involve questioning or interviewing, listening to students' responses, and carefully observing their written work. As previously mentioned, questions carefully prepared in advance can be of major help in generating the data about students' thinking that the teacher needs to proceed with a lesson in a productive manner. Observing students' written work while they are working on a task can also foster useful insights into students' thinking and into misconceptions they may hold. Given that time for observation is short, teachers should not feel compelled to examine the work of every student during the class period. Indeed, more benefit may be gained by looking at just a few papers, but each in some depth. Through experience, a teacher gains insight into what to look for and how to effectively deal with specific student responses that arise. Although successful methods of carefully attending to students' thinking and students' work are difficult to communicate, the value of these activities has been shown in several programs of research. For further discussion of listening to students, see the chapter by Driscoll in this volume.

After the Lesson

After a lesson has been taught, a teacher should take time to reflect on the lesson, thinking about how subsequent lessons can build on, and be linked with, the current lesson and previous lessons. The ideas learned in one lesson must be reviewed, extended, and linked with the ideas from other lessons. Connecting lessons in this way promotes students' learning—learning that can be represented as a meaningful network of ideas. Forming such a network of connected ideas is essential in developing a deep knowledge of mathematics (Hiebert and Carpenter 1992). In addition to logically connecting ideas among lessons, a teacher needs to consider students' work as part of the process of planning future lessons.

Examining and Taking Account of Students' Work

Students' thinking must be studied while a lesson is being taught, and students' written work should be carefully examined after a lesson has been taught. But what should be the focus of such assessments? The most obvious things to look for are students' misconceptions, error patterns, and other observations that fall under the umbrella of shortcomings. Students' strengths, however, are equally important to observe. These strengths might be in such areas as the ability to represent ideas geometrically or symbolically, to systematically organize ideas in table or graphic form, or to carry out complex procedures orderly and accurately. Although this type of assessment takes time, the teacher should remember that not every assignment must be graded. Similarly, when looking for data on students' knowledge alone, the teacher need not examine every student's paper.

The results of assessing students' work can be taken into account in several ways. First, establishing a good sense of students' thinking and students' knowledge provides the basis for choosing tasks that have a high cognitive demand but are still within students' reach. Second, specific knowledge about students can be used to structure appropriate questions for students that will stimulate discussion and draw out important connections among mathematical ideas, including ideas that may be in place but need to be made explicit, or ideas that need to be more fully developed. Finally, a working sense of the scope and depth of students' knowledge can be used to structure homework assignments that help overcome deficiencies and keep important skills and procedures at a high proficiency level. For further discussion of assessing students' learning and other important outcomes, see the chapter by Ziebarth in this volume.

A Final Note

The decisions a teacher makes when planning and implementing a problem-solving approach to mathematics instruction have a pronounced impact on the progress students make toward acquiring mathematical proficiency. If students are going to realize their mathematical potential, their teachers must be much more than "guides on the side"; they must be active, thoughtful decision makers prior to instruction, while teaching, and after teaching a lesson.

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